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1992 J. Phys. A: Math. Gen. 25 L673

(http://iopscience.iop.org/0305-4470/25/11/010)

J. Phys. A: Math. Gen. 25 (1992) L673–L677. Printed in the UK

LETTER TO THE EDITOR

Hydrodynamical reductions of the lattice KP hierarchy

B A Kupershmidt

The University of Tennessee Space Institute, Tullahoma, TN 37388, USA

Received 25 February 1992

Abstract. Hydrodynamic reductions are found for the lattice KP hierarchy and its zero-dispersion limit. This is similar to the continuous case where the reductions result in the dispersive water waves and Benney hierarchies respectively.

The continuous KP hierarchy [1] is the system of Lax equations

$$\mathscr{L}_{,t} = [(\mathscr{L}^m)_+, \mathscr{L}] \qquad m \in \mathbb{N} \tag{1}$$

with the Lax operator

$$\mathscr{L} = \xi + \sum_{i=0}^{\infty} \xi^{-i-1} A_i \qquad \xi = \partial = \partial/\partial x \qquad A_i = A_i(x, t).$$

In the quasiclassical (=zero-dispersion) limit [2] one gets the so-called Benney hierarchy [3]

$$\bar{\mathscr{L}}_{,t} = \{(\bar{\mathscr{L}}^m)_+, \bar{\mathscr{L}}\} \qquad \bar{\mathscr{L}} = p + \sum_{i \geq 0} A_i p^{-i-1} \tag{2}$$

with the Poisson bracket $\{ , \}$ being the standard one:

$$\{f, g\} = f_{,p} g_{,x} - f_{,x} g_{,p}.$$

For the case $m = 2$, one gets from (2) the Benney system proper [4]:

$$\tfrac{1}{2} A_{i,t} = A_{i+1,x} + i A_{i-1} A_{0,x} \qquad i \in \mathbb{Z}_+ \tag{3}$$

which results upon taking the moments

$$A_i = \int_0^h u^i \, dy \qquad u = u(x, y, t) \qquad h = h(x, t) \tag{4}$$

of the free-surface long wave $(2+1)$-dimensional hydrodynamical system [4]

$$\begin{cases} \tfrac{1}{2} u_{,t} = u u_{,x} + h_{,x} - u_{,y} \int_0^y u_{,x} \, dy \\ \tfrac{1}{2} h_{,t} = \left(\int_0^h u \, dy \right)_{,x}. \end{cases}$$

For all other m, one has similar $(2+1)$-dimensional hydrodynamical representations of the Benney hierarchy (2) [3].

In the case when the velocity u is y-independent, the moment map (4) becomes

$$A_i = h u^i \qquad [= A_0 (A_1/A_0)^i] \tag{5}$$

the Lax operator $\bar{\mathscr{L}}$ (2) becomes

$$\tilde{\mathscr{L}} = p + \sum_{i\geq 0} hu^i p^{-i-1} = p + h/(p-u) \tag{6}$$

and one arrives at an *a priori* non-obvious conclusion that the infinite system (3) and its higher analogues (2) have two-component hydrodynamical reductions (which, moreover, are Hamiltonian [3]). The full KP hierarchy (1), of which the Benney hierarchy (2) is the quasiclassical limit, also has a hydrodynamical reduction

$$A_i = h(\partial + u)^i(1) \tag{7}$$

which is also Hamiltonian [5]. For $m = 2$, one gets the so-called dispersive water waves (DWW) system [5]:

$$\begin{cases} u_{,t} = (u^2 + 2h - u_{,x})_{,x} \\ h_{,t} = (2uh + h_{,x})_{,x}. \end{cases}$$

The reduction (5) is, thus, the zero-dispersion limit of the dispersive reduction (7).

The purpose of this letter is to work out hydrodynamical reductions of the lattice KP hierarchy [6, 7]

$$L_{,t} = [(L^m)_+, L] \qquad m \in \mathbb{N} \tag{8}$$

with the Lax operator

$$L = \zeta + \sum_{i=0}^{\infty} a_i \zeta^{-i}.$$

Here ζ is the operator version of Δ, Δ itself being the dual to the shift:

$$\zeta^s f = \Delta^s(f)\zeta^s$$

$$(\Delta^s f)(n) = f(n+s) \qquad (\Delta^s f)(x) = f(x + s\Delta x) \qquad n, s \in \mathbb{Z}.$$

For the first flow $m = 1$, the Lax equation (8) is

$$a_{i,t} = (\Delta - 1)(a_{i+1}) + a_i(1 - \Delta^{-i})(a_0) \qquad i \in \mathbb{Z}_+. \tag{9}$$

In the quasiclassical limit $\Delta = \exp(\varepsilon\partial)$, one gets from (9)

$$a_{i,t} = a_{i+1,x} + i a_i a_{0,x} \qquad i \in \mathbb{Z}_+. \tag{10}$$

It is easy to check that the latter system allows the hydrodynamical reduction

$$a_i = hu^i \qquad i \in \mathbb{Z}_+ \tag{11}$$

$$\begin{cases} u_{,t} = u(u+h)_{,x} \\ h_{,t} = (uh)_{,x}. \end{cases} \tag{12}$$

Similarly, the full system (9) is found, after some experimentation, to have the hydrodynamical reduction

$$a_i = h \prod_{r=0}^{i} u^{(-r)}/u^{(-i)} \qquad i \in \mathbb{Z}_+ \tag{13}$$

$$\begin{cases} u_{,t} = u(1 - \Delta^{-1})(u+h) \\ h_{,t} = (\Delta - 1)(uh). \end{cases} \tag{14}$$

Clearly, the quasiclassical limit of formulae (13), (14) yields formulae (11), (12) respectively.

What is the general fact applicable to the *whole hierarchy* (8) and to its quasiclassical limit

$$\bar{L}_{,t}=\{(\bar{L}^m)_+,\bar{L}\}_{(1)} \qquad \bar{L}=p+\sum_{i=0}^{\infty}a_ip^{-i}$$
$$\{f,g\}_{(1)}=p(f_{,p}g_{,x}-f_{,x}g_{,p}) \tag{15}$$

which, for $m=1$, reduces to the formulae (11)-(14)? Let us start with the simpler zero-dispersion case first. The Hamiltonian structure $B=(B_{ij})$ of the lattice Lax hierarchy (8) is [6, 7]

$$B_{ij}=\Delta^j a_{i+j}-a_{i+j}\Delta^{-i}. \tag{16}$$

This means that the motion equation (8) can be recast into the form

$$a_{i,t}=\sum_j B_{ij}\left(\frac{\delta H}{\delta a_j}\right)$$

with

$$H=H_{m+1}=\frac{1}{m+1}\operatorname{Res}(L^{m+1})$$

Res being the operation singling-out the ζ^0-term. In the zero-dispersion limit, the Hamiltonian matrix (16) becomes

$$\bar{B}_{ij}=\mathrm{i}a_{i+j}\partial+\partial ja_{i+j}. \tag{17}$$

The Hamiltonian matrix (17) is of the type associated with Poisson manifolds [8]. By formula (38) in [8], the reduction map (11) is a Hamiltonian map between the Hamiltonian matrices (17) and

$$\bar{B}=\begin{pmatrix}0 & u\partial\\ \partial u & 0\end{pmatrix}. \tag{18}$$

In other words, the reduction map (11) is self-consistent and converts the quasiclassical limit (15) of the lattice KP hierarchy into the integrable commuting Hamiltonian hierarchy

$$u_{,t}=u\partial(\delta\tilde{H}/\delta h) \qquad h_{,t}=\partial(u\delta\tilde{H}/\delta u)$$

where

$$\tilde{H}=\frac{1}{m+1}\operatorname{Res}(\tilde{L}^{m+1})$$
$$\tilde{L}=p+\sum_{i\geqslant 0}hu^ip^{-i}=p+hp/(p-u). \tag{19}$$

We can now handle the fully discrete case. Let

$$\tilde{B}=\begin{pmatrix}0 & u(1-\Delta^{-1})\\ (\Delta-1)u & 0\end{pmatrix} \tag{20}$$

be a Hamiltonian matrix in the (u, h) space. Then the map (13) is a Hamiltonian map between the Hamiltonian structures $\tilde{B}$ (20) and B (16).

Proof. Let J be the Fréchet derivative of the map (13). We have to check that [7]

$$J\tilde{B}J^{\dagger}=B$$

where $J^{\dagger}$ is the adjoint of J. In long hand, we have to verify that

$$\frac{\mathrm{D}a_i}{\mathrm{D}h}(\Delta-1)u\left(\frac{\mathrm{D}a_j}{\mathrm{D}u}\right)^{\dagger}+\frac{\mathrm{D}a_i}{\mathrm{D}u}u(1-\Delta^{-1})\left(\frac{\mathrm{D}a_j}{\mathrm{D}h}\right)^{\dagger}=\Delta^j a_{i+j}-a_{i+j}\Delta^{-1} \tag{21}$$

where it is understood that a_i's are expressed through u and h. Using the identities

$$\frac{\mathrm{D}a_i}{\mathrm{D}u}=a_i\frac{1-\Delta^{-i}}{1-\Delta^{-1}}\frac{1}{u} \qquad \frac{\mathrm{D}a_i}{\mathrm{D}h}=a_i\frac{1}{h} \tag{22}$$

the equality (21) becomes

$$a_i(h^{-1}\Delta^j-\Delta^{-i}h^{-1})a_j=\Delta^j a_{i+j}-a_{i+j}\Delta^{-i}$$

which decomposes into the pair of equivalent identities

$$(a_i/h)^{(-j)}a_j=a_{i+j} \tag{23a}$$

$$a_i(a_j/h)^{(-i)}=a_{i+j}. \tag{23b}$$

But

$$\begin{aligned}(a_i/h)^{(-j)}a_j\ &[\text{by (13)}]\\ &=\left[\prod_{r=0}^{i}u^{(-r)}/u^{(-i)}\right]^{(-j)}h\prod_{l=0}^{j}u^{(-l)}/u^{(-j)}\\ &=h\prod_{r=j}^{i+j}u^{-r}[u^{(-i-j)}]^{-1}\prod_{l=0}^{j}u^{(-l)}/u^{(-j)}\\ &=h\prod_{r=0}^{i+j}u^{(-r)}/u^{(-i-j)}=a_{i+j}\end{aligned}$$

which is (23a). □

Thus, the hierarchy of lattice hydrodynamical Lax equation

$$\hat{L}_{,t}=[(\hat{L}^m)_+,\hat{L}] \qquad m\in\mathbb{N} \tag{24}$$

$$\begin{aligned}\hat{L}&=\zeta+\sum_{i=0}^{\infty}h\prod_{r=0}^{i}u^{(-r)}[u^{(-i)}]^{-1}\zeta^{-i}\\ &=\zeta+h\sum_{i=0}^{\infty}(u\zeta^{-1})^i\\ &=\zeta+h(1-u\zeta^{-1})^{-1}\end{aligned} \tag{25}$$

is an integrable hierarchy with the Hamiltonian structure

$$u_{,t}=u(1-\Delta^{-1})(\delta\hat{H}/\delta h) \qquad h_{,t}=(\Delta-1)(u\delta\hat{H}/\delta u)$$

$$\hat{H}=\frac{1}{m+1}\operatorname{Res}[(\hat{L})^{m+1}].$$

In particular, for $m=1$,

$$\hat{H}=\tfrac{1}{2}\operatorname{Res}\hat{L}^2=\tfrac{1}{2}\operatorname{Res}(\zeta+h+hu\zeta^{-1}+\ldots)^2\sim\frac{h^2}{2}+uh$$

and we recover the system (14).

Remark. The Hamiltonian matrix $\tilde{B}$ (20) of the hydrodynamical reduction of the lattice KP hierarchy is identical to the first Hamiltonian structure of the Toda lattice hierarchy under the identification

$$u = a_1 \qquad h = a_0.$$

One gets the Hamiltonian Toda matrix as the 2×2 submatrix $0 \leqslant i,\ j \leqslant 1$ of the Hamiltonian matrix (16) upon letting $\{a_r = 0 | \forall r > 1\}$.

Remark. One can show that the hydrodynamical reduction (13) is unique in the class of formulae

$$a_i = h^{(\mu i)} \prod_{r=0}^{i} u^{(-r\nu)}/u^{(-\nu i)}. \tag{26}$$

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